1.d Find the generators of the group $(\mathbb{Z}_8,+)$

Ans: Here, $(\mathbb{Z}_8,+)$ is a Cyclic group. and $\mathbb{Z}_8 = \{\bar{0},\bar{1},\bar{2},\bar{3},\bar{4},\bar{5},\bar{6},\bar{7}\}$

Now, $\gcd(k,8) = 1$, where $k = 1, 3, 5, 7$

Therefore, the generators of $\mathbb{Z}_8$ are $\bar{1}, \bar{3}, \bar{5}, \bar{7}$.

f. Prove that every group of prime order is cyclic.

Ans: Let, G be a group of prime order P (say)

Since, $P>1$, G has an element $a \neq e$.

Let, H be the subgroup generated by a. i.e. $H = \{a^n : n \in \mathbb{Z}\} = [a]$.

Now by Lagrange's Theorem, $|[a]|$ divides P

Hence, $|[a]| = 1$ or P.

Since, $a \neq e$., Therefore $|[a]| \neq 1$. and so $|[a]| = P$.

Now $[a] \in G$ and $|[a]| = |G| = P$.

Hence, $G = [a]$. This shows that every group of prime order is cyclic.

c. Let, G be a group and $a \in G$. Prove that the mapping $f_a : G \to G$, defined by $f_a(x) = ax$, $\forall x \in G$, is a bijection.

Ans: Let, $x, y \in G$. Then $f_a(x) = ax$ and $f_a(y) = ay$

Now, $f_a(x) = f_a(y) \Rightarrow ax = ay \Rightarrow x = y$ [by l.c.l]

For $x \neq y \Rightarrow ax \neq ay \Rightarrow f_a(x) \neq f_a(y)$.

i.e. f_a is one-one.

Let, P be an arbitrary element in G. of a codomain set has a pre-image $x \in G$ (say), then $f_a(x) = P$. $\Rightarrow ax = P \Rightarrow x = a^{-1}P$, which exist in G. Therefore f_a is on-to.

Hence, f_a is bijective.

b. Give an example of an abelian group which is not cyclic.

Ans: $G = (\mathbb{Q},+)$ is abelian but not Cyclic.

If possible let $(\mathbb{Q},+)$ is a Cyclic group, generated by an element $a \in \mathbb{Q}$ $a \neq 0$.

Therefore, $(\mathbb{Q},+)$ is a Cyclic group generated by a, then every element can be expressed as na, where n is some integer.

Here, $\frac{1}{2}a \in \mathbb{Q}$ but $\frac{1}{2}a$ can-not be expressed as na, $n \in \mathbb{Z}$.

We arrive a contradiction.

$\therefore (\mathbb{Q},+)$ is not a Cyclic group.

2. b.(i) Show that $Z(G)$, the centre of a group G is a normal subgroup of G.

Ans: 1st we show that $Z(G)$ is a subgroup of G.

Since, $e \in Z(G)$, $\therefore Z(G) \neq 0$.

Let, $a, b \in Z(G)$. then $ag = ga$ and $bg = gb \Rightarrow gb^{-1} = b^{-1}g$, $\forall g \in G$.

Hence, $(ab^{-1})g = a(b^{-1}g) = a(gb^{-1}) = (ga)b^{-1} = g(ab^{-1})$, $\forall g \in G$.

$\Rightarrow ab^{-1} \in Z(G)$.

So, $Z(G)$ is a subgroup of G.

Now for $g \in G$ and $a \in Z(G)$.

$gag^{-1} = agg^{-1} = a \in Z(G)$.

~~and Hence,~~ $\cancel{gZ(G)g^{-1} \in Z(G)}$.

Hence, $Z(G)$ is a normal subgroup of G.

2.b.(ii) Let $G = S_3$ and G' be the multiplicative group $(\{1, -1\}, .)$, define $\phi : G \to G'$ by $\phi(a) = 1$, if a is an even permutation.
$= -1$, if a is an odd permutation.

Prove that ϕ is a surjective homomorphism. Also find $\ker\phi$ and hence determine a normal subgroup of S_3.

Ans: Let, $\alpha, \beta \in S_3$ then $\alpha\beta \in S_3$

Case. I. Let, α, β be both even then $\alpha\beta$ is even and $\phi(\alpha\beta) = 1$
$= 1.1$
$= \phi(\alpha).\phi(\beta)$

case. II. Let, α, β be both odd then $\alpha\beta$ is even and

$\phi(\alpha) = -1$, $\phi(\beta) = -1$ and $\phi(\alpha\beta) = 1. = -1.-1 = \phi(\alpha).\phi(\beta)$.

case. III. If α be even and β be odd then $\alpha\beta$ be odd.

$\phi(\alpha\beta) = -1 = 1.-1 = \phi(\alpha).\phi(\beta)$.

$\therefore$ In all cases we see that $\phi(\alpha\beta) = \phi(\alpha).\phi(\beta)$, $\forall \alpha, \beta \in S_3$.

i.e. ϕ is a Hommomorphism.

Since, every element of the codomain set G' has a pre-image in the domain set G.

$\therefore \phi$ is onto.

ϕ is a homomorphism, here 1 is the identity element of G' and $\ker\phi = \{\alpha \in S_3 : \phi(\alpha) = 1\}$.

$\therefore \ker\phi$ contains all even permutation of S_3.

$\therefore \ker\phi = A_3$

Since, $\ker\phi$ is a normal sub-group of S_3, it follows that A_3 is a normal subgroup of S_3.

(iii) Prove that a finite integral domain is a field.

Ans: Let, D be a finite I.D.

Then D is a commutative ring with unity $1.(\neq 0)$

Now we have to show that D is a field.

i.e. if $a \in D$, where $a \neq 0$ then a has a multiplicative inverse in D.

$\because$ D is a finite set, we write the element of D as $a_1, a_2, \ldots, a_n$.

Now consider the elements $aa_1, aa_2, \ldots, aa_n$

Here all these elements are distincts for otherwise $aa_i = aa_j \Rightarrow a_i = a_j$, which is not true.

$\because$ all these elements $\in D$. and D has exactly n elements, they must be the elements $a_1, a_2, \ldots, a_n$ in some order.

Consequently, one of them is 1. $(\because 1 \in D)$.

So, $a_i a = aa_i = 1$, for some i.

Thus a has a multiplicative inverse in D.

$\therefore$ It follows that D is a field.

d.(i) Prove that a finite group is isomorphic to a subgroup of some permutation group.

Ans: Let, $G = \{g_1, g_2, \ldots, g_n\}$.

Let, g be an arbitrary element of G., then the element $gg_1, gg_2, \ldots, gg_n \in G$. and no two of these elements are equal, otherwise if $gg_i = gg_j \Rightarrow g_i = g_j$, which is not the case.

$\therefore \begin{pmatrix} g_1 & g_2 & \cdots & g_n \\ gg_1 & gg_2 & \cdots & gg_n \end{pmatrix}$ is a permutation on the set $\{g_1, g_2, \ldots, g_n\}$

We denote this permutation π_g.

Let, S_n be the set of all permutations ~~of~~ on the set $\{g_1, g_2, \ldots, g_n\}$

Let us define a mapping $\phi : G \to S_n$ by $\phi(g_i) = \pi_{g_i}$

i.e. $\phi(g_i) = \begin{pmatrix} g_1 & g_2 & \cdots & g_n \\ g_i g_1 & g_i g_2 & \cdots & g_i g_n \end{pmatrix}$., $\forall i = 1, 2, \ldots, n$.

We now show that ϕ is a homomorphism.

Let, $g_i, g_j \in G$. , then $g_i g_j \in G$. ~~and $\phi(g_i g_j) =$~~

and $\phi(g_i g_j) = \begin{pmatrix} g_1 & g_2 & \cdots & g_n \\ g_i g_j g_1 & g_i g_j g_2 & \cdots & g_i g_j g_n \end{pmatrix}$.

$$= \begin{pmatrix} g_1 & g_2 & \cdots & g_n \\ g_i g_1 & g_i g_2 & \cdots & g_i g_n \end{pmatrix} \circ \begin{pmatrix} g_1 & g_2 & \cdots & g_n \\ g_j g_1 & g_j g_2 & \cdots & g_j g_n \end{pmatrix}$$

$$= \phi(g_i) \circ \phi(g_j).$$

$\therefore \phi$ is a homomorphism.

For $x \in G$, $x \in \operatorname{Ker}\phi \Leftrightarrow \phi(x) = \begin{pmatrix} g_1 & g_2 & \cdots & g_n \\ g_1 & g_2 & \cdots & g_n \end{pmatrix}$.

$$\Leftrightarrow \begin{pmatrix} g_1 & g_2 & \cdots & g_n \\ x g_1 & x g_2 & \cdots & x g_n \end{pmatrix} = \begin{pmatrix} g_1 & g_2 & \cdots & g_n \\ g_1 & g_2 & \cdots & g_n \end{pmatrix}.$$

$\Leftrightarrow x g_k = g_k$, for $k = 1, 2, \ldots, n$.

$\Leftrightarrow x = e_G$

$\therefore \operatorname{Ker}\phi = \{e_G\}$

$\therefore \phi$ is one-one, $\therefore \phi$ is a monomorphism.

Also, $\phi(G)$ is a subgroup of S_n, $\therefore G \simeq \phi(G)$.

i.e. G is isomorphic to a subgroup of S_n.